Test Sets for Vertex Cover Problems

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Abstract

We describe the structure of the unique minimal test set T for a family of vertex cover problems. The set T corresponds to the Gröbner basis of the binomial ideal for the problem as described in [1]. While T has a surprisingly simple structure, in particular when the underlying graph is complete, it is $\mathcal{NP}$-complete to decide whether the test set contains an improving element for a given feasible solution in the case of a complete graph

1 Introduction

In [1] Conti and Traverso pointed out that the Buchberger algorithm may serve as a solution strategy for solving families of integer programming problems with varying right-hand side. It computes the unique minimal test set for a given family of integer programming problems and, thus, can be used to iteratively improve a feasible solution until optimality is reached.

Unfortunately, test sets tend to be large in general and their computation may be very time and space consuming. Thus, we became interested in finding a combinatorial characterization of test sets, e.g. if the underlying family of problems shares a common combinatorial structure.

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We give a first result of this type. We will show that the Gröbner test set of the weighted vertex cover problem has a surprisingly simple structure. In the next section we will introduce the problems, present the structural results in Section 3 and apply it to the special cases of complete and complete bipartite graphs. In Section 4 we give the $\mathcal{NP}$-completeness result and discuss the consequences in Section 5. Our notation is fairly standard as in e.g. [4].

2 Basic Definitions

The weighted vertex cover problem is a generalization of the well-known vertex cover problem [4]. Let $G = (V, E)$ be a graph with vertex set V of size n and edge set E of size m. Let $N(U)$ denote the set of neighbors of a set $U \subseteq V$, i.e. the set of vertices in $V \setminus U$ that have a neighbor in U. Given integers b_e on each edge $e \in E$, a vertex cover consists of integer weights x_v on the vertices of G such that each edge e is covered at least b_e times, i.e. $x_u + x_v \geq b_e$ for all edges $e = \{u, v\}$. Note, that if the underlying graph G is bipartite, finding the vertex cover of minimum total weight is the linear programming dual to finding a maximum weight matching in G.

For an integer programming formulation, let $A = (a_{ev})_{e\in E, v\in V}$ denote the transpose of the vertex-edge-incidence matrix of G, i.e. we have $a_{ev} = 1$ if and only if v is incident to e, and $a_{ev} = 0$ otherwise. Given a vector $b \in \mathbb{Z}_+^m$, a feasible vertex cover is a vector $x \in \mathbb{Z}_+^n$ on the vertices of G such that each edge $e \in E$ is covered at least b_e times, i.e. $x_u + x_v \geq b_e$ for all edges $e = \{u, v\}$. In order to formulate the corresponding extended integer linear program we introduce slack variables y_e for $e \in E$. Then the vertex cover problem reads as follows:

$$\begin{aligned} \min \quad & \mathbf{1}^t x \\ \text{s.t.} \quad & Ax - Iy = b \\ & x \in \mathbb{Z}_+^n,\ y \in \mathbb{Z}_+^m, \end{aligned} \tag{1}$$

where $I \in \mathbb{Z}_+^{m\times m}$ is the identity matrix.

The objective function induces a natural ordering on the vertex covers. For the Gröbner basis approach possible ties need to be broken. Hence, let $\prec$ denote a lexicographical elimination order [3] on $\mathbb{Z}_+^{n+m}$ such that $\chi(v) \prec \chi(e)$ for the characteristic vectors in $\mathbb{Z}_+^{n+m}$ of any $v \in V$ and $e \in E$. Then the refined order

$\prec'$ is defined as

$$x \prec' y \Longleftrightarrow \begin{cases} \sum_{v\in V} x_v < \sum_{v\in V} y_v, \text{ or} \\ \sum_{v\in V} x_v = \sum_{v\in V} y_v, \text{ and } x \prec y \end{cases}$$

Instead of (1) we will study the following IP-formulation of the weighted vertex cover problem:

$$VC(b) \quad \min_{\prec'} : Ax - Iy = b, \quad x \in \mathbb{Z}_+^n,\ x' \in \mathbb{Z}_+^m, \tag{2}$$

where $\min_{\prec'}$ means minimizing with respect to the refined order $\prec'$.

Now a set $T \subseteq \mathbb{Z}_+^{n+m}$ is a test set for the family $(VC(b))_{b\in\mathbb{Z}_+^m}$ if given a feasible non-optimal solution (x, x') to some program $VC(b)$ there exists $(t, t') \in T$ such that $(x, x') - (t, t')$ is feasible for the same program and has a smaller objective value than (x, x') w.r.t. $\prec'$. There exists a unique minimal test set for this family which corresponds to the Gröbner basis of a certain binomial ideal [1,7].

3 Structure of the minimal test set

The minimal test set for the family $(VC(b))_{b\in\mathbb{Z}_+^m}$ consists of three classes of elements. The first class contains vectors that leave the total weight of the cover unchanged but improve with respect to $\prec$. The second class of vectors improves the weight by 1 but leads to a lexicographical increase. Finally, the third class of elements improves the weight by 1, improves with respect to $\prec$, and satisfies an additional technical condition, which becomes considerably simpler when we consider complete graphs.

Theorem 1 *Let $G = (V, E)$ be a graph and let A be the transpose of the corresponding vertex-edge-incidence matrix. Furthermore, let $b \in \mathbb{Z}_+^m$, and $\prec'$ a lexicographical elimination order on $\mathbb{Z}_+^{n+m}$ defined as above.*
Let $T \subseteq \{(t, t') \in \mathbb{Z}_+^{n+m} \mid At - t' = 0\ , t \in \{0, \pm 1\}^n\}$ be the set of vectors additionally satisfying one of the following three conditions:

there exist vertex sets $U \subseteq V$ and $W \subseteq N(U)$, where $N(U)$ denotes the set of vertices adjacent to U, such that $t = \chi(U) - \chi(W)$
and U is inclusionwise minimal satisfying either
$\sum_{v\in V} t_v = 0$ and $0 \prec (t, t')$, or $\sum_{v\in V} t_v = 1$ and $(t, t') \prec 0$, or
$\sum_{v\in V} t_v = 1$ and $0 \prec (t, t')$ and there is no $v_0 \in V \setminus W$ such that
$\sum_{v\in V\setminus v_0} t_v = 0$ and $0 \prec (t - \chi(v_0), t' - A\chi(v_0))$.

Then T is the unique minimal test set for the family $(VC(b))_{b\in\mathbb{Z}_+^m}$ of integer programs.

In the following, we consider two special cases of the theorem. If G is a complete graph we get a substantially easier description of the minimal test set because of the trivial adjacency relations between vertices in G:

Example 2 *Let G be a complete graph, and let $T_{complete} \subseteq \{(t,t') \in \mathbb{Z}_+^{n+m} \mid At - t' = 0,\ t \in \{0,\pm 1\}^n\}$ be the set of vectors additionally satisfying one of the following three conditions:*

$\sum_{v\in V} t_v = 0$ *and* $0 \prec' (t,t')$ *, or* $\sum_{v\in V} t_v = 1$ *and* $(t,t') \prec' 0$ *, or*
$t = \chi(v)$ *for the minimal* $v \in V$ *with respect to* $\prec'$ *.*

Then $T_{complete}$ is the unique minimal test set for the family $(VC(b))_{b\in\mathbb{Z}_+^m}$ of integer programs.

If G is a complete bipartite graph the structure of the minimal test set and even its size varies with the choice of the refining order. Let V_1 and V_2 be the bipartition of the vertex set of G. As an example, we consider an order $\prec$ on $\mathbb{Z}_+^{n+m}$ such that $\chi(v_1) \prec \chi(v_2)$ for all $v_i \in V_i,\ i = 1, 2$, and furthermore there is a vertex $u \in V_1$ which is incident to the n first edges in G with respect to $\prec$.

Example 3 *Let G be a complete bipartite graph, and let $T_{bipartite} \subseteq \{(t,t') \in \mathbb{Z}_+^{n+m} \mid At - t' = 0,\ t \in \{0,\pm 1\}^n\}$ be the set of vectors additionally satisfying one of the following conditions:*

there exist subsets $U \subseteq V_1,\ W \subseteq V_2$ *such that*

$t = \chi(U) - \chi(W),\ \sum_{v\in V} t_v = 0,$ *and* $u \in U,$ *or*
$t = \chi(W) - \chi(U),\ \sum_{v\in V} t_v = 0,$ *and* $u \notin U,$ *or*
$t = \chi(U) - \chi(W),\ \sum_{v\in V} t_v = 1,$ *and* $u \notin U,$ *or*
$t = \chi(W) - \chi(U),\ \sum_{v\in V} t_v = 1,$ *and* $u \in U,$ *or*
if $|V_1| = |V_2|,$ *then* $t = \chi(V_2) - \chi(V_1)$.

Then $T_{bipartite}$ is the unique minimal test set for the family $(VC(b))_{b\in\mathbb{Z}_+^m}$ of integer programs.

4 Complexity

As finding a minimum vertex cover is $\mathcal{N}P$-complete but the structure of the test set is simple, it is obvious that chosing an improving element must be difficult in general.

Theorem 4 *For the first two classes of elements of $T_{complete}$ it is $\mathcal{N}P$-complete to decide whether, given a graph $G = (V,E)$ and a feasible solution $(x,x') \in$*

$\mathbb{Z}_+^{n+m}$ *to some* $VC(b)$*, they contain an improving element* (t, t')*.*

Proof: Reduction from VERTEX COVER [4].

In the case of bipartite graphs matching techniques can be used to find improvements in polynomial time.

5 Conclusion

We have given a combinatorial description of the Gröbner test set of a family of weighted vertex cover problems. This set has a particularly simple structure in the case of complete and complete bipartite graphs. While the vertex cover problem in the former case is $\mathcal{N}P$-complete it is polynomially solvable in the latter. The structure of the test set does not seem to reflect this difference.

References

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